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Nuclear structure corrections to gyromagnetic factor of the bound lepton *

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In the framework of the quasipotential method the covariant expression for the two-particle vertex operator is obtained. The nuclear structure corrections of orders $(Z\alpha)^4$, $(Z\alpha)^5$ including recoil effects to gyromagnetic factors of the bound electron and muon are calculated. Numerical value of the contribution of order $(Z\alpha)^5$ is obtained by means of the dipole parameterization for the nuclear charge form factor in the range of the nuclear charges $Z = 7 \div 32$.

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I. INTRODUCTION

Experimental and theoretical investigations of the g-factor of a free and bound leptons (the electron and muon) are important both for the check of the Standard Model and determination more correct values of several fundamental parameters of the theory including the lepton-to-proton mass ratio [1, 2]. The growth of the interest to theoretical calculations on this problem in the last years is connected with new experiments where the hydrogen-like ions with moderate values of the nuclear charge Z are used [3] (see experimental data shown in Table I). An increase of the experimental accuracy for the electron g-factor measurements in such simple atoms generates a need for the calculation of new corrections which can become essential for the comparison of the theory and experiment.

Gyromagnetic factor of the bound lepton can be represented in the form:

$$g_l = 2 + \Delta g_{rel} + \Delta g_{rad} + \Delta g_{rec} + \Delta g_{str} + ... \tag{1}$$

Relativistic corrections Δg_{rel}, radiative corrections Δg_{rad}, recoil corrections Δg_{rec} were investigated in Refs.[4, 5, 6, 7, 8, 9, 10, 11, 12, 13] with an accuracy up to fourth order terms. Certain contributions to the g_l of higher order over α related for the most part to the vacuum polarization and the lepton self energy were studied in Refs.[13, 14, 15]. An agreement between theoretical and experimental results in the problem of the gyromagnetic factor of the bound electron is observed to the present. For hydrogenic ions with sufficiently high values Z the nuclear structure effects Δg_{str} acquire important role. They can be taken

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TABLE I: Experimental data and theoretical values on the g-factors of the bound electron.

Ratio of the g-factors	Experiment	Ref.	Theory [1, 3]
$\frac{g_{e^-}(H)}{g_{e^-}}$	$1-17.709(13)\times 10^{-6}$	[18]	$1-17.7053\times 10^{-6}$
$\frac{g_{e^-}(D)}{g_{e^-}}$	—	—	$1-17.7125\times 10^{-6}$
$\frac{g_{e^-}(Mu)}{g_{e^-}}$	—	—	$1-17.591\times 10^{-6}$
$\frac{g_{e^-}(H)}{g_{e^-}(D)}$	$1+7.22(3)\times 10^{-9}$	[19]	$1+7.247\times 10^{-9}$
$\frac{g_{e^-}(H)}{g_{e^-}(T)}$	$1+10.7(1.5)\times 10^{-9}$	[20]	$1+10.7\times 10^{-9}$
$\frac{g_{e^-}(^4He^+)}{g_{e^-}}$	$1-70.87(30)\times 10^{-6}$	[21]	$1-70.87\times 10^{-6}$
$\frac{g_{e^-}(^{12}C^{5+})}{2}$	$1+520.798(2)\times 10^{-6}$	[22]	$1+520.796\times 10^{-6}$
$\frac{g_{e^-}(^{16}O^{7+})}{2}$	$1+23.514(2)\times 10^{-6}$	[23]	$1+23.511\times 10^{-6}$

into account by electromagnetic form factors of minimal multipolarity describing the distributions of the electric charge and magnetic moment. Main contribution of order $(Z\alpha)^4$ to the Δg_{str} is determined by the nuclear charge radius R_N which is the differential parameter of the electric charge distribution. It was originally obtained on the basis of nonrelativistic approach in Ref.[13] and by means of the Dirac equation in Ref.[16] where relativistic correction of order $(Z\alpha)^6 m_1^2 R_N^2$ to the Δg_{str} was found also. The quasipotential method for the problem of magnetic moment of hydrogenic atom was formulated in Refs.[4, 5]. In this approach the two-particle vertex function describing the interaction of the bound state with external electromagnetic field was introduced. Its expansion by the perturbative series for hydrogenic systems allows to calculate radiative, recoil corrections of different orders. In the case of the constituent particles with arbitrary spin the construction of the vertex operator in two main orders by the perturbative series was carried out in Ref.[17]. The aim of present work consists in the calculation of the nuclear structure effects of orders $(Z\alpha)^4$ and $(Z\alpha)^5$ including recoil effects in the g-factors of the bound electron and muon on the basis of the quasipotential method. Considering that in recent experiments with hydrogenic ions an accuracy comprises several ppb [18, 19, 20, 21, 22, 23] it is evident that the role of the nuclear structure corrections in the lepton g-factor for moderate and high Z will be enhance.

II. GENERAL FORMALISM

Magnetic moment of the two-particle bound state can be written as follows [4, 5]:

$$\mathbf{M} = -\frac{i}{2}\left[\frac{\partial}{\partial \mathbf{\Delta}} \times <\Psi_{n,Q}|\mathbf{J}(0)|\Psi_{n,P}>\right], \quad \mathbf{\Delta} = \mathbf{Q} - \mathbf{P}, \tag{2}$$

where the matrix element of electromagnetic current operator $\mathbf{J}$ between the bound states with total momenta $\mathbf{Q}$ and $\mathbf{P}$ respectively can be expressed in terms of the wave functions of the bound system $\Psi_{n,Q}(\mathbf{q})$, $\Psi_{n,P}(\mathbf{p})$ and generalized vertex function Γ_μ represented in Fig.1

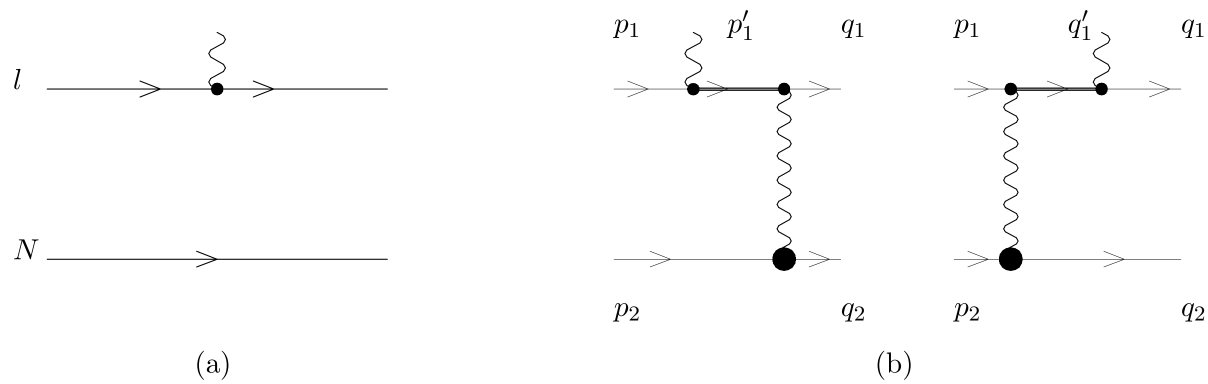

FIG. 1: Generalized two-particle vertex function Γ_μ: the diagram (a) represents $\Gamma^{(0)}_\mu$ with free noninteracting particles, the diagrams (b) determine the nucleus (N) structure corrections of order $(Z\alpha)^4$ in the g-factor of the bound lepton (l). Bold line denotes negative energy part of the lepton propagator. Large bold circle denotes the interaction vertex of the nucleus with electromagnetic field proportional to the factor $(F_1(k^2)-1)$.

in the form:

$$<\Psi_{n,Q}|J_\mu(0)|\Psi_{n,P}>=\int\frac{d\mathbf{p}d\mathbf{q}}{(2\pi)^6}\bar\Psi_{n,Q}(\mathbf{q})\Gamma_\mu(\mathbf{p},\mathbf{q})\Psi_{n,P}(\mathbf{p}), \tag{3}$$

The two-particle vertex function Γ_μ is determined in terms of the five-point Green-like function

$$R_\mu=<0|\psi_1(t,\mathbf{x_1})\psi_2(t,\mathbf{x_2})J_\mu(0)\bar\psi_1(\tau,\mathbf{y_1})\bar\psi_2(\tau,\mathbf{y_2})|0>, \tag{4}$$

projected on the positive energy states [4, 5]:

$$\Gamma_\mu=G^{-1}R^{(+)}G^{-1},\quad R^{(+)}_\mu=\bar u_1\bar u_2R_\mu u_1u_2, \tag{5}$$

where G is the two-particle Green function and u_1, u_2 are the wave functions of a free particles. For loosely bound system in quantum electrodynamics all introduced quantities Γ, G^{-1}, R can be represented by the perturbative series:

$$\Gamma=\Gamma^{(0)}+\Gamma^{(1)}+\Gamma^{(2)}+...,R=R_0+R_1+R_2+...,G^{-1}=G_0^{-1}-V_1-.... \tag{6}$$

Substituting Eq.(6) in Eq.(5) we obtain the following relations for generalized vertex function:

$$\Gamma^{(0)}=G_0^{-1}R_0G_0^{-1},\Gamma^{(1)}=G_0^{-1}R_1G_0^{-1}-V_1G_0\Gamma^{(0)}-\Gamma^{(0)}G_0V_1, \tag{7}$$

$$\Gamma^{(2)}=G_0^{-1}R_2G_0^{-1}-V_2G_0\Gamma^{(0)}-\Gamma^{(0)}G_0V_2-V_1G_0\Gamma^{(1)}-\Gamma^{(1)}G_0V_1-2V_1G_0\Gamma^{(0)}G_0V_1,$$

where G_0 is the Green function of two noninteracting particles, V_1 is the quasipotential of the one-photon interaction, V_2 is the quasipotential of the two-photon interaction.

The transformation law for the wave function $\Psi_{n,P}(\mathbf{p})$ of two bound particles with spins s_1, s_2 from the rest frame to the reference frame moving with momentum $\mathbf{P}$ was obtained in Ref.[24] in the form:

$$\Psi_{n,P}(\mathbf{p})=D_1^{s_1}(R^W_{L_P})D_2^{s_2}(R^W_{L_P})\Psi_{n,0}(\mathbf{p}), \tag{8}$$

where $D^s(R)$ is the rotation matrix, R^W is the Wigner rotation and L_P is the Lorentz boost from the rest frame to the reference frame moving with momentum $\mathbf{P}$. The rotation matrix can be expressed in terms of the Lorentz transformation matrices as follows:

$$D^s(R^W_{L_P}) = S^{-1}(\mathbf{p}_{1,2})S(\mathbf{P})S(\mathbf{p}). \tag{9}$$

The quasipotential bound state wave function $\Psi_0(\mathbf{p})$ in the rest frame of the composite system satisfies the quasipotential equation [25]:

$$G_0^{-1}\Psi \equiv \left(\frac{b^2}{2\mu_R} - \frac{\mathbf{p}^2}{2\mu_R}\right)\Psi_0(\mathbf{p}) = \int V(\mathbf{p},\mathbf{q},M)\Psi_0(\mathbf{q})\frac{d\mathbf{q}}{(2\pi)^3}, \tag{10}$$

where μ_R is relativistic reduced mass:

$$\mu_R = \frac{E_1E_2}{M} = \frac{M^4-(m_1^2-m_2^2)^2}{4M^3}, E_{1,2} = \frac{M^2-m_{2,1}^2+m_{1,2}^2}{2M}, \tag{11}$$

$M = E_1 + E_2$ is the mass of the bound state,

$$b^2(M) = \frac{[M^2-(m_1+m_2)^2][M^2-(m_1-m_2)^2]}{4M^2}. \tag{12}$$

In the nonrelativistic limit the equation (10) reduces to the Shroedinger equation with the Coulomb potential. The main contribution to the vertex function Γ_μ is equal

$$\Gamma_\mu^{(0)\lambda\sigma,\rho\omega}(\mathbf{p},\mathbf{q}) = \bar{u}_1^\lambda(\mathbf{q_1})\gamma_\mu u_1^\rho(\mathbf{p_1})\bar{u}_2^\sigma(\mathbf{q_2})u_2^\omega(\mathbf{p_2})(2\pi)^3\delta(\mathbf{p_2}-\mathbf{q_2})\delta^{\sigma\omega}, \lambda,\sigma,\rho,\omega = \pm\frac{1}{2}, \tag{13}$$

where the Dirac spinors are

$$u^\lambda(p) = \sqrt{\frac{\epsilon(p)+m}{2\epsilon(p)}}\begin{pmatrix} 1 \\ \dfrac{(\boldsymbol{\sigma}\mathbf{p})}{\epsilon(p)+m}\end{pmatrix}\chi^\lambda, \quad \lambda = \pm\frac{1}{2}. \tag{14}$$

Relativistic four momenta of the particles in the initial and final states are defined as follows:

$$p_{1,2} = \varepsilon_{1,2}(\mathbf{p})v \pm \sum_{i=1}^3 n^{(i)}(v)p^i, \quad v = \frac{P}{M}, \tag{15}$$

$$q_{1,2} = \varepsilon_{1,2}(\mathbf{q})v' \pm \sum_{i=1}^3 n^{(i)}(v')q^i, \quad v' = \frac{Q}{M},$$

and $n^{(i)}$ are three four vectors defined by

$$n^{(i)}(v) = \left\{v^i, \delta^{ij} + \frac{1}{1+v^0}v^iv^j\right\}. \tag{16}$$

Using relations (8)-(9) we can transform the matrix element (3) to the form containing the product of two traces over the spinor indices of both particles. For this aim the following relations are useful:

$$S_{\alpha\beta}(\Lambda)u_\beta^\lambda(\mathbf{p}) = \sum_{\sigma=-s}^{s} u_\alpha^\sigma(\Lambda\mathbf{p})D^s_{\sigma\lambda}(R^W_{\Lambda p}), \tag{17}$$

$$\bar{u}^\lambda_\beta(\mathbf{p})S^{-1}_{\beta\alpha}(\Lambda)=\sum_{\sigma=-s}^{s}D^{+,S}_{\lambda\sigma}(R^W_{\Lambda p})\bar{u}^\sigma_\alpha(\Lambda\mathbf{p}).$$

Substituting expressions (8), (9), (13) in Eq.(3) and using relations (17) we obtain:

$$<\Psi_{n,Q}|J^{(0)}_\mu|\Psi_{n,P}>=\int\frac{d\mathbf{p}d\mathbf{q}}{(2\pi)^3}\bar{\Psi}_0(\mathbf{q})\bar{u}_1(0)\frac{(\hat{q}_1+m_1)}{\sqrt{2\epsilon_1(q)(\epsilon_1(q)+m_1)}}S^{-1}(L_Q)\gamma_\mu S(L_P)\times \tag{18}$$

$$\times\frac{(\hat{p}_1+m_1)}{\sqrt{2\epsilon_1(p)+m_1)}}u_1(0)\bar{u}_2(0)\frac{(\hat{q}_2+m_2)}{\sqrt{2\epsilon_2(q)+m_2)}}S^{-1}(L_Q)\times$$

$$\times S(L_P)\frac{(p_2+m_2)}{2\epsilon_2(p)(\epsilon_2(p)+m_2)}u_2(0)\delta(\mathbf{p_2}-\mathbf{q_2})\Psi_0(\mathbf{p}),$$

where

$$p_1=(\epsilon_1(p),\mathbf{p}),p_2=(\epsilon_2(p),-\mathbf{p}),q_1=(\epsilon_1(q),\mathbf{q}),q_2=(\epsilon_2(q),-\mathbf{q}).$$

Introducing the one-particle projectors on the states with definite spins $s_{1,2}$ in the rest frame

$$\hat{\pi}_{1,2}=[u_{1,2}(0)\bar{u}_{1,2}(0)]=\frac{(1+\gamma^0)}{2}\frac{(1+\gamma^5\hat{s}_{1,2})}{2}, \tag{19}$$

we can represent basic matrix element (18) in the form:

$$<\Psi_{n,Q}|J^{(0)}_\mu(0)|\Psi_{n,P}>=\int\frac{d\mathbf{p}d\mathbf{q}}{(2\pi)^3}\bar{\Psi}_0(\mathbf{q})\Psi_0(\mathbf{p})\delta(\mathbf{p_2}-\mathbf{q_2})\times \tag{20}$$

$$\times Tr\left[\frac{(\hat{q}_1+m_1)}{\sqrt{2\epsilon_1(q)(\epsilon_1(q)+m_1)}}S^{-1}(L_Q)\gamma_\mu S(L_P)\frac{(\hat{p}_1+m_1)}{\sqrt{2\epsilon_1(p)(\epsilon_1(p)+m_1)}}\frac{(1+\gamma^0)(1+\gamma^5\hat{s}_1)}{4}\right]\times$$

$$\times Tr\left[\frac{(\hat{q}_2+m_2)}{\sqrt{2\epsilon_2(q)(\epsilon_2(q)+m_2)}}S^{-1}(L_Q)S(L_P)\frac{(\hat{p}_2+m_2)}{\sqrt{2\epsilon_2(p)(\epsilon_2(p)+m_2)}}\frac{(1+\gamma^0)(1+\gamma^5\hat{s}_2)}{4}\right].$$

In the next orders of the perturbative series the matrix element of electromagnetic current keeps the general structure (20) but the form of the vertex function (5) will be changed. The expression (20) is useful tool for the calculation of the different order corrections to the lepton gyromagnetic factors by means of the computer systems of analytical calculations. The system Form [26] is employed in the calculation of the correction Δg_{str}.

III. NUCLEAR STRUCTURE CORRECTIONS OF ORDER $(Z\alpha)^4$

In the framework of the quasipotential method formulated in the section 2 there are two sources of the nuclear structure contributions of order $(Z\alpha)^4$ to the gyromagnetic factors of the bound particles. The first contribution is represented on the diagrams Fig.1 (b). The interaction vertex of the second particle (nucleus) with electromagnetic field (large bold circle) is determined by two form factors $F_{1,2}$.

Let us consider the contributions of the diagrams Fig.1 (b). The trace over the lepton spinor indices for the first diagram Fig.1 (b) entering in Eq. (20) takes the form:

$$T_{11}=\frac{1}{64m_1^2}Tr\Big\{(1+\gamma^0)(1+\gamma^5\hat{s}_1)(\hat{q}_1+m_1)\gamma^0\left[\hat{v}'+1-(1-\gamma^0)\right]\gamma^\nu\times \tag{21}$$

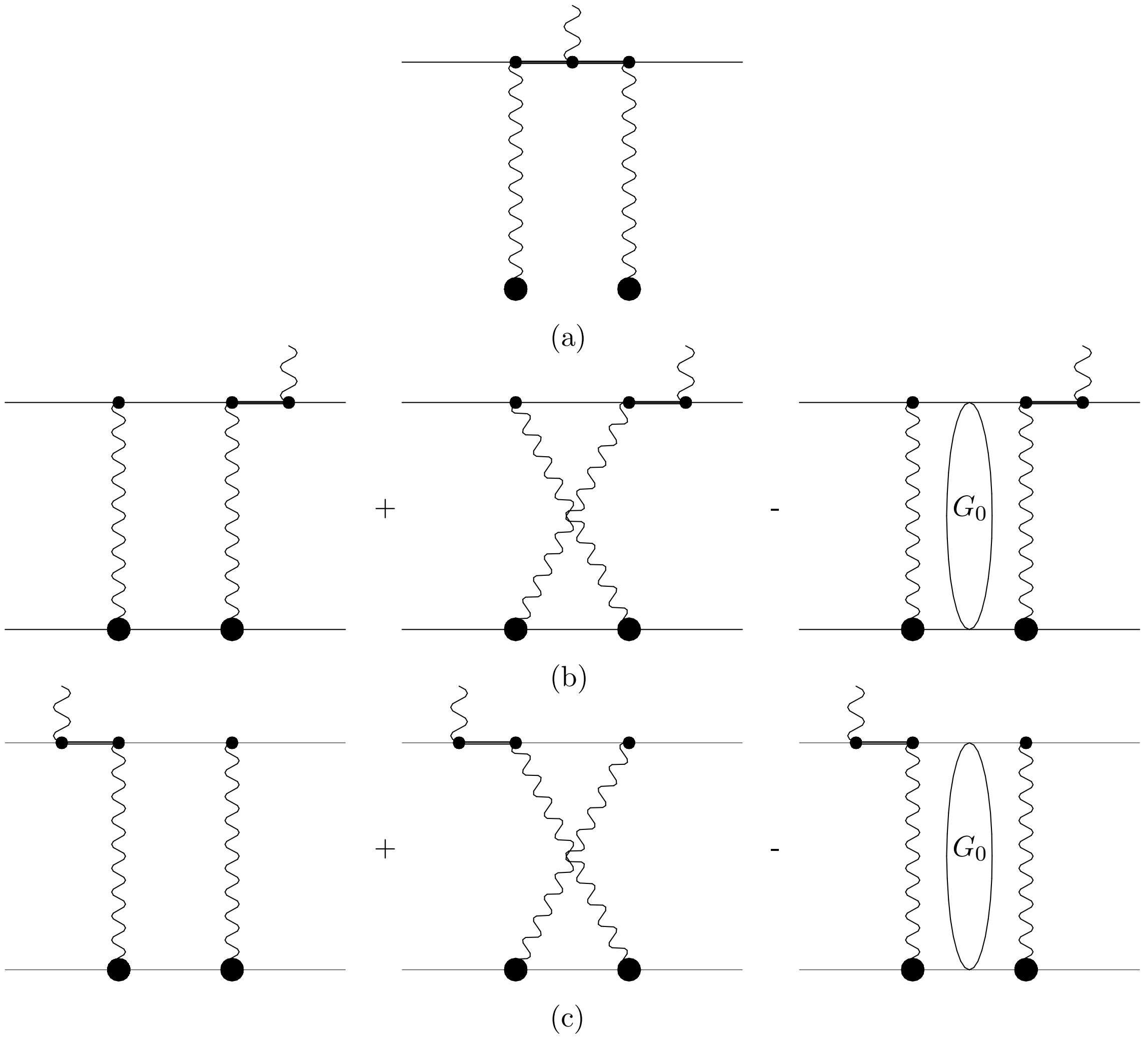

FIG. 2: Generalized two-particle vertex function Γ_μ: the nuclear structure corrections of order $(Z\alpha)^5$. Bold line denotes the negative energy part of the lepton propagator.

$$\times\Lambda^-(\mathbf{p_1'})\gamma^0\gamma^\mu\left[\hat v+1-(1-\gamma^0)\right]\gamma^0(\hat p_1+m_1)\Big\},$$

$$\Lambda^-(\mathbf{p_1'})=\frac{1}{2m_1}\gamma^0\left[m_1(\gamma^0-1)+\hat p+\hat\Delta\left(1-\frac{m_1}{2M}\right)\right],\mathbf{p}_1'=\mathbf{p}_1+\mathbf{\Delta},\ p=(0,\mathbf{p}). \qquad (22)$$

Similar contribution of the second diagram Fig.1 (b) is equal

$$T_{11}'=\frac{1}{64m_1^2}Tr\Big\{(1+\gamma^0)(1+\gamma^5\hat s_1)(\hat q_1+m_1)\gamma^0\left[\hat v'+1-(1-\gamma^0)\right]\gamma^\mu\times \qquad (23)$$

$$\times\Lambda^-(\mathbf{q_1'})\gamma^0\gamma^\nu\left[\hat v+1-(1-\gamma^0)\right]\gamma^0(\hat p_1+m_1)\Big\},$$

$$\Lambda^-(\mathbf{q_1'})=\frac{1}{2m_1}\gamma^0\left[m_1(\gamma^0-1)+\hat q-\hat\Delta\left(1-\frac{m_1}{2M}\right)\right],\mathbf{q}_1'=\mathbf{q}_1-\mathbf{\Delta},\ q=(0,\mathbf{q}). \qquad (24)$$

The trace over the spinor indices of the second particle (nucleus) which is equal for both diagrams Fig. 1 (b) has the following structure:

$$T_{21}=\frac{1}{64m_2^2}Tr\Bigl\{(1+\gamma^0)(1+\gamma^5\hat s_2)(\hat q_2'+m_2)\gamma^0\left[\hat v'+1-(1-\gamma^0)\right]\gamma^0\times \tag{25}$$
$$\times\left[\gamma^\nu F_1+\frac{i}{2m_2}F_2\sigma^{\nu\lambda}k_\lambda\right]\left[\hat v+1-(1-\gamma^0)\right](\hat p_2'+m_2)\Bigr\}.$$

After the contraction of the expressions (21) and (23) with (25) over the Lorentz index ν ($\mu\to i$ is the vector index) we obtain that the sum of the trace products can be written in the form:

$$[\boldsymbol{\sigma}_1\times\boldsymbol{\Delta}]F_1\left(1-\frac{m_1}{m_1+m_2}\right)+[\boldsymbol{\sigma}_2\times\boldsymbol{\Delta}]\,(F_1+2F_2)\,\frac{m_1}{m_1+m_2}. \tag{26}$$

Substituting sequentially Eq.(26) in Eq.(20) and then in Eqs.(3) and (2) and calculating the $rot_{\boldsymbol{\Delta}}$ we keep only the contributions to the lepton gyromagnetic factor (the particle 1). The nuclear structure correction takes the form ($\mathbf{s_1}\to<\boldsymbol{\sigma}_1>$):

$$\Delta g_1^{str}=\frac{4}{3}m_1^2R_N^2(Z\alpha)^4\left(1-4\frac{m_1}{m_2}\right), \tag{27}$$

where the following expansion of the Dirac form factor F_1 at small $\mathbf{k}^2$ is used:

$$F_1(\mathbf{k}^2)=1-\frac{\mathbf{k}^2}{6}R_N^2. \tag{28}$$

The expression (27) contains not only the correction of order $(Z\alpha)^4$ but also the recoil effects of the first order over the ratio m_1/m_2. The same order correction as (27) appears in the second order of the perturbation theory. The contribution of the second order of the perturbative series to the matrix element (3) has the form:

$$2<\Psi_n|\mathbf{J}(0)|\delta\Psi_n>=2<\Psi_n|\mathbf{J}\tilde G\Delta V|\Psi_n>=2\sum_m\frac{<\Psi_n|\mathbf{J}(0)|\Psi_m><\Psi_m|\Delta V|\Psi_n>}{E_n-E_m}, \tag{29}$$

where Ψ_n are the Coulomb wave functions of Eq.(10). The reduced Coulomb Green function (RCGF) $\tilde G$ doesn't contain the free two-particle Green function because its contribution cancels the term which appears from the half of last iteration contribution $\Gamma^{(2)}$ in Eq.(7). The nuclear structure effects in Eq.(29) are determined by the terms of the quasipotential $\Delta V\sim R_N^2$ as it follows from Eq. (28). To calculate the matrix element in Eq.(29) we need to know the form of relativistic correction of order $\mathbf{p}^2/m_1^2$ including recoil effects in the vertex operator Γ_μ which was found in Refs.[4, 5]. Accounting the form of the $1S$ state RCGF in the coordinate representation [27]

$$\tilde G_{1S}(r,0)=\frac{Z\alpha\mu^2}{4\pi}\frac{2e^{-x/2}}{x}\left[2x(\ln x+C)+x^2-5x\right], \tag{30}$$

(C=0.5772156649... is the Euler constant) we can express necessary nuclear structure correction in the g-factor of the bound lepton in the second order of the perturbative series as follows:

$$\Delta g_2^{str}=-\frac{8}{3}R_N^2\pi Z\alpha\int\Psi_{1S}(\mathbf{r})\frac{\mathbf{p}^2}{3m_1^2}\left(1+\frac{m_1}{2m_2}\right)d\mathbf{r}\tilde G_{1S}(r,0)|\Psi_n(0)|= \tag{31}$$

TABLE II: Nuclear structure corrections in the g-factor of the bound electron and muon of orders $(Z\alpha)^4$ and $(Z\alpha)^5$.

Z	$R_N(fm)$	$b_{NS}[\times 10^{-9}]$ [13]	electron Δg^{str}		muon Δg^{str}	
			$(Z\alpha)^4[\times 10^{-9}]$	$(Z\alpha)^5[\times 10^{-12}]$	$(Z\alpha)^4[\times 10^{-4}]$	$(Z\alpha)^5[\times 10^{-6}]$
7	2.54	0.79	0.79	0.13	0.33	-0.17
8	2.737	1.56	1.56	0.26	0.65	-0.42
9	2.90	2.80	2.80	0.53	1.17	-0.89
10	2.992	4.54	4.54	0.95	1.90	-1.66
12	3.08	9.98	9.97	2.49	4.19	-4.54
15	3.191	26.14	26.13	8.10	11.03	-15.50
18	3.423	62.37	62.37	22.80	26.39	-47.88
24	3.643	223.29	223.28	117.13	94.68	-244.35
32	4.088	888.63	888.60	550.02	377.57	-1461.07

$$= \frac{4}{3} R_N^2 m_1^2 (Z\alpha)^4 \left(1 - \frac{7m_1}{2m_2}\right).$$

Summing two corrections (27) and (31) we obtain total contribution of the nuclear structure effects of order $(Z\alpha)^4$ with the account recoil correction:

$$\Delta g^{str}_{l,(Z\alpha)^4} = \frac{8}{3} R_N^2 m_1^2 (Z\alpha)^4 \left(1 - \frac{15m_1}{4m_2}\right). \tag{32}$$

Numerical values (32) for several hydrogenic ions are presented in Table II.

IV. SPIN DEPENDENT NUCLEAR STRUCTURE CORRECTIONS OF ORDER $(Z\alpha)^4$

In general case, the matrix element of the electromagnetic current for the particle of arbitrary spin S is determined by means of $(2S+1)$ form factors (charge, magnetic, quadruple, et al.). When the magnetic moment of simple atomic systems is studied it may be possible to take into account the form factors of the minimal multipolarity describing the distributions of the electric charge and magnetic moment. The one-particle matrix element J_μ of the electromagnetic current operator between states with momenta p and q can be written as follows:

$$J_\mu = \bar{U}(\mathbf{p}) \left\{ \Gamma_\mu F_1^D + \frac{1}{2m} \Sigma_{\mu\nu} k^\nu F_2^P \right\} U(\mathbf{q}). \tag{33}$$

The wave function $U(\mathbf{p})$ of a particle with arbitrary spin entering in Eq.(33) can be presented in the form (see, for instance [28, 29]):

$$U = \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \xi^{\alpha_1\alpha_2\ldots\alpha_p}_{\dot\beta_1\dot\beta_2\ldots\dot\beta_q} \\ \eta^{\beta_1\beta_2\ldots\beta_q}_{\dot\alpha_1\dot\alpha_2\ldots\dot\alpha_p} \end{pmatrix}, \quad p+q = 2S, \tag{34}$$

where the spin-tensors ξ, η are symmetrical in upper and lower indices. For the particle of half integer spin p=S+1/2, q=S-1/2. In the case of integer spin p=q=S. The Lorentz transformation of the spinors ξ and η can be written in the form [29, 30]:

$$\xi = \exp\left(\frac{\mathbf{\Sigma}\boldsymbol{\delta}}{2}\right)\xi_0, \quad \eta = \exp\left(-\frac{\mathbf{\Sigma}\boldsymbol{\delta}}{2}\right)\xi_0, \tag{35}$$

where the direction of the vector $\boldsymbol{\delta}$ coincides with the velocity of the particle, $th\delta = v$. The generator of the Lorentz transformation $\mathbf{\Sigma}$ is equal:

$$\mathbf{\Sigma} = \sum_{i=1}^{p}\boldsymbol{\sigma}_i - \sum_{i=p+1}^{p+q}\boldsymbol{\sigma}_i, \tag{36}$$

and $\boldsymbol{\sigma}_i$ acts on the ith index of the spinor ξ_0 as follows:

$$\boldsymbol{\sigma}_i\xi_0 = (\boldsymbol{\sigma}_i)_{\alpha_i\beta_i}(\xi_0)_{\ldots\beta_i\ldots}. \tag{37}$$

In the standard representation, which is introduced in analogy with the spin 1/2, the free particle wave function (34) can be written with the accuracy $(v/c)^2$ in the form:

$$U(\mathbf{p}) = \begin{pmatrix} \left[1+\frac{(\mathbf{\Sigma}\mathbf{p})^2}{8m^2}\right]\xi_0 \\ \frac{\mathbf{\Sigma}\mathbf{p}}{2m}\xi_0 \end{pmatrix}. \tag{38}$$

The components of the matrix $\Sigma_{\mu\nu}$ in (33) are the generators of the boosts and rotations [29, 30]:

$$\Sigma_{n0} = \begin{pmatrix} \Sigma_n & 0 \\ 0 & -\Sigma_n \end{pmatrix}, \quad \Sigma_{mn} = -2i\epsilon_{mnk}\begin{pmatrix} s_k & 0 \\ 0 & s_k \end{pmatrix}, \quad \mathbf{s} = \frac{1}{2}\sum_{i=1}^{2S}\boldsymbol{\sigma}_i. \tag{39}$$

Considering the contribution of the diagrams Fig.1 (b) to the matrix element (20) let us point out that with desired accuracy the bispinor contraction over line of the second particle (nucleus)

$$\bar{U}_2(\mathbf{q}_2)\mathcal{B}_2U_2(\mathbf{p}_2)F_1(k^2) \approx \left(1-\frac{1}{6}k^2R_N^2\right) \tag{40}$$

(where the matrix $\mathcal{B}_2$ is the generalization for the matrix β_2 used in the case of the spin 1/2 particles) doesn't contain any other additional factors connected with the nuclear spin. So, the form of the correction (27) remains unchanged with the account the first order recoil effect. But the correction of the second order over the lepton-to-proton mass ratio contains the spin dependent terms. Indeed, using Eqs.(33), (38)-(39) we have the following bispinor contraction for both particles (see the diagrams Fig.1 (b) where the nucleus interacts with external electromagnetic field):

$$\frac{e_2}{2m_2}\bar{u}_1(\mathbf{q}_1)\alpha_1^iu_1(\mathbf{p}_1)F_1(k^2)\bar{U}_2(\mathbf{q}_2)\mathcal{B}_2\left[\mathcal{A}_2^i\frac{1}{2}\left(1-\mathcal{B}_2\right)\mathbf{A}_2+\mathbf{A}_2\frac{1}{2}\left(1-\mathcal{B}_2\right)\mathcal{A}_2^i\right]U(\mathbf{p}_2) \approx$$

$$\approx -\frac{Ze}{2m_2}\frac{m_1}{m_1+m_2}i\left[\boldsymbol{\sigma}_1\times\mathbf{\Delta}\right]K_{S_2}\frac{1}{6}\mathbf{k}^2R_N^2, \tag{41}$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & \mathbf{\Sigma} \\ \mathbf{\Sigma} & 0 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{42}$$

TABLE III: Spin dependent nuclear structure corrections in the g-factor of the bound electron and muon of order $(Z\alpha)^4$: $\Delta g_l^{str} = \frac{2}{3}R_N^2 m_1^2 (Z\alpha)^4 \frac{m_1^2}{m_2^2}\left[6K_{S_2}+Z+29\right]$.

Z	$R_N(fm)$	s_2	electron $[\times 10^{-16}]$	muon $[\times 10^{-6}]$
7	2.54	1/2	0.12	0.02
		1	0.13	0.02
8	2.737	0	0.17	0.03
		5/2	0.27	0.05
11	2.94	3/2	0.53	0.10
		3	0.63	0.12
		4	0.71	0.13
13	3.035	5/2	0.94	0.17
		3	0.97	0.18
24	3.643	7/2	5.28	0.97

$$K_{s_2} = \frac{<\boldsymbol{\Sigma}_2^2>}{3} = \begin{cases} \frac{4s_2}{3}, & s_2 \text{ is integer nuclear spin} \\ \frac{4s_2+1}{3}, & s_2 \text{ is half integer nuclear spin} \end{cases}. \quad (43)$$

The following commutation relations are useful:

$$[\Sigma_i, \Sigma_j] = 4i\epsilon_{ijk}s_k, \quad [\Sigma_i, s_j] = i\epsilon_{ijk}\Sigma_k. \quad (44)$$

Other part of the contribution of order $(Z\alpha)^4 m_1^2/m_2^2$ is determined by relativistic effects in the vertex operator $\Gamma^{(0)}$ which were obtained in Ref.[17] on the basis of relations (34)-(39) in the form:

$$\Delta\Gamma^{(0)}_{rel} = \frac{e_1}{2m_1}\boldsymbol{\sigma}_1\left\{-\frac{\mathbf{p}^2}{3m_1^2}\left(1+\frac{m_1}{2m_2}\right)+\frac{\mathbf{p}^2}{m_2^2}\left[1-K_{S_2}\left(1+\frac{Z}{3}\right)-\frac{Z-1}{6}\right]\right\}. \quad (45)$$

Calculating the contribution of the second order of the perturbative series by means of Eqs.(29) and (44) and taking into account the relation (41) we obtain the following total value of the nuclear structure correction:

$$\Delta g^{str}_{l,total} = \frac{8}{3}m_1^2 R_N^2 (Z\alpha)^4\left\{1-\frac{15m_1}{4m_2}+\frac{m_1^2}{4m_2^2}\left[6K_{S_2}+Z+29\right]\right\}. \quad (46)$$

It is helpful to remark that two terms in Eqs.(41) and (45) proportional to $K_{s_2}\cdot Z$ will be mutually cancelled in the correction (46). The expression (46) is the generalization of Eq.(32) for the case of the nuclear spin dependent contribution which contains also the squared ratio of the particle masses m_1/m_2. Numerical values of the third addendum in Eq.(46) for different Z and s_2 are compared with the correction of order $(Z\alpha)^6 m_1^2 R_N^2$ [16] and represented in Table III.

V. NUCLEAR STRUCTURE CORRECTIONS OF ORDER $(Z\alpha)^5$

The nuclear structure corrections of order $(Z\alpha)^5$ in the g-factor of the bound lepton are specified by the vertex function $\Gamma^{(2)}$ in Eq.(7) and shown in Fig.2. There is need to take

into account that the corrections of such order contain also additional small factor m_1/Λ (Λ is the form factor parameter) which lead to the decrease this contribution as compared with Eq.(32). Nevertheless just as for the g-factor of the bound electron in hydrogenic ion with sufficiently high Z, so for gyromagnetic factor of the bound muon the value of this correction can increase essentially and become important for modern experiments.

The two-photon exchange diagrams relevant to the $\Gamma^{(2)}$ can be divided into two parts. In the Feynman amplitudes of the first part the interaction of the lepton with external field occures between two exchange interactions. We consider these two-photon exchanges in the limit of static nucleus neglecting recoil corrections when the nucleus interacts with the photons via the Dirac form factor F_1. The interaction of the lepton with two exchanged Coulomb photons is plotted in Fig. 2 (a). The bispinor contraction over the lepton line with consideration for the interaction terms (7) incorporates negative energy parts of the lepton propagators shown in Fig.2 (a) by bold line. The trace over the spinor indices of the first particle entering in Eq. (20) can be written as follows:

$$T_{12}=\frac{1}{16}Tr\Big\{(1+\gamma^0)(1+\gamma^5\hat{s}_1)(\hat{q}_1+m_1)[\hat{v}'+1-(1-\gamma^0)]\gamma^0\times \tag{47}$$

$$\times\left[\epsilon_1(\mathbf{k}+\boldsymbol{\Delta})-k^0+\gamma^0(\hat{k}+\hat{\Delta}-m_1)\right]\gamma^0\gamma^\lambda\left[\epsilon_1(\mathbf{k})-k^0+\gamma^0(\hat{k}-m_1)\right][\hat{v}+1-(1-\gamma^0)](\hat{p}_1+m_1)\Big\}$$

$$\approx 4m_1^2(\epsilon_1(k)-m_1)[\boldsymbol{\sigma}_1\times\boldsymbol{\Delta}].$$

Then we can write the contribution of the diagram Fig.2 (a) to the g-factor of the bound lepton in the form ($\epsilon_1(k)=\sqrt{k^2+m_1^2}$):

$$\Delta g^{(1)}_{l,(Z\alpha)^5}=-\frac{29m_1\mu^3(Z\alpha)^5}{32\pi}\int_0^\infty\frac{dk}{k^2}\frac{(F_1^2-1)(\epsilon_1(k)-m_1)}{(\epsilon_1(k)+m_1)^2\epsilon_1(k)^2}. \tag{48}$$

The scale of the integration momenta in Eq.(48) is of order of the lepton mass. In spite of the infrared finiteness of the appeared expression in the leading order over m_1/m_2 we make the subtraction of the point-like nuclear contribution in Eq.(48). To obtain correct energy spectrum in the case of point-like nucleus it is necessary to consider the bound state effects and keep the relative motion momenta of the particles $\mathbf{p}$, $\mathbf{q}$ (see the discussion, for example, in Ref.[31]). Another part of the two-photon exchange diagrams which lead to the nuclear structure corrections of order $(Z\alpha)^5$ is presented in Fig.2 (b), (c). The trace calculations over the spinor indices and the contraction over the Lorentz indices can be performed for these diagrams in the same manner as for previous contributions. To illustrate the general structure of intermediate expressions in this case we write the trace product regarding to the first diagram in Fig.2 (b):

$$Tr\Big\{(1+\gamma^0)(1+\gamma^5\hat{s}_1)(\hat{q}_1+m_1)(\hat{v}'+1)\gamma^\lambda\Lambda^-(-\boldsymbol{\Delta})\gamma^0\gamma^\mu(\hat{p}_1+\hat{k}+m_1)\gamma^\nu(\hat{v}+1)(\hat{p}_1+m_1)\Big\}\times \tag{49}$$

$$\times Tr\Big\{(1+\gamma^0)(1+\gamma^5\hat{s}_2)(\hat{q}_2+m_2)(\hat{v}'+1)\gamma^\mu(m_2\gamma^0-\frac{1}{2}\hat{\Delta}-\hat{k}+m_2)\gamma^\nu(\hat{v}+1)(\hat{p}_2+m_2)\Big\}F_1(k)F_1(k+\Delta).$$

Total contribution of all diagrams in Fig. 2 (b), (c) in the trace product of Eq.(20) is the following:

$$m_2F_1^2\Big\{-\frac{50}{3}k_0^4[\boldsymbol{\sigma}_1\times\boldsymbol{\Delta}]-32m_1^2k_0^2[\boldsymbol{\sigma}_1\times\mathbf{k}]-\frac{32}{3}k^4[\boldsymbol{\sigma}_1\times\boldsymbol{\Delta}]+ \tag{50}$$

$$+8k^4[\boldsymbol{\sigma}_1\times\mathbf{k}]-\frac{14}{3}k^2k_0^2[\boldsymbol{\sigma}_1\times\boldsymbol{\Delta}]+16k^2k_0^2[\boldsymbol{\sigma}_1\times\mathbf{k}]\Big\}.$$

But contrary to earlier considered expressions for the matrix elements of electromagnetic current (2) which are linear with respect to $[\boldsymbol{\sigma}_1\times\boldsymbol{\Delta}]$ these diagrams involve also the integral of the form $\int d^4k[\boldsymbol{\sigma}_1\times\mathbf{k}]F(k,\Delta)$. Its calculation includes the shift of the integration variable $\mathbf{k}\to\mathbf{k}-\frac{\boldsymbol{\Delta}}{2}$ and the calculation of the $rot_{\boldsymbol{\Delta}}$ from the integration function. After rotating the k^0 contour of the Feynman loop integration we integrate over four dimensional Euclidean space using the relation

$$\int d^4k=4\pi\int k^3dk\int_0^\pi\sin^2\phi d\phi,\quad k^0=k\cos\phi. \tag{51}$$

The integration over the angle variable ϕ can be done analytically. After that we can present the sum of the contributions shown in Fig.2 (b), (c) to the lepton g-factor in the form:

$$\Delta g^{(2)}_{l,(Z\alpha)^5}=-\frac{4}{3}\frac{m_2\mu^3(Z\alpha)^5}{\pi}\int_0^\infty\frac{dk}{k}\times \tag{52}$$

$$\times\left\{(F_1^2-1)\left[25I_3+16I_1+16k^2I_4+31I_2+8k^2\frac{d}{dm_1^2}(I_1-I_2)\right]-16F_1'(0)\frac{\mu}{m_1m_2k}\right\},$$

$$I_1=\frac{1}{\pi}\int_{-1}^1\frac{\sqrt{1-x^2}dx}{(k^2+4m_1^2x^2)(k^2+4m_2^2x^2)}=\frac{\omega_2-\omega_1}{4k^3(m_2^2-m_1^2)},\omega_i=\sqrt{k^2+4m_i^2}, \tag{53}$$

$$I_2=\frac{1}{\pi}\int_{-1}^1\frac{x^2\sqrt{1-x^2}dx}{(k^2+4m_1^2x^2)(k^2+4m_2^2x^2)}=\frac{k(m_1^2-m_2^2)+m_2^2\omega_1-m_1^2\omega_2}{16k(m_2^2-m_1^2)m_1^2m_2^2}, \tag{54}$$

$$I_3=\frac{1}{\pi}\int_{-1}^1\frac{x^4\sqrt{1-x^2}dx}{(k^2+4m_1^2x^2)(k^2+4m_2^2x^2)}=\frac{2m_1^2m_2^2(m_1^2-m_2^2)+k^2(m_1^4-m_2^4)+k(m_2^4\omega_1-m_1^4\omega_2)}{64m_1^4m_2^4(m_1^2-m_2^2)}, \tag{55}$$

$$I_4=\frac{1}{\pi}\int_{-1}^1\frac{(1-x^2)^{3/2}dx}{(k^2+4m_1^2x^2)^2(k^2+4m_2^2x^2)}= \tag{56}$$

$$=\frac{1}{16k^5\omega_1(m_1^2-m_2^2)^2}\left[-k^4+k^2(6m_2^2-2m_1^2+\omega_1\omega_2)+4(2m_1^4-6m_1^2m_2^2+m_2^2\omega_1\omega_2)\right].$$

The contribution of the iteration term in the vertex operator $\Gamma^{(2)}$ (7) was taken into account in Eq.(52). We make also necessary subtraction of the point-like nuclear contribution in Eq.(52) because in the low momentum region there exists logarithmic infrared divergence. To obtain numerical results on the basis of the expressions (48) and (52) the dipole parameterization for the Dirac form factor F_1 is used:

$$F_1(k^2)=\frac{1}{\left(1+\frac{k^2}{\Lambda^2}\right)^2},\quad\Lambda=\frac{\sqrt{12}}{R_N}. \tag{57}$$

The total contributions of order $(Z\alpha)^5$ to the Δg_{str} for the electron and muon presented in Table II have different signs because the corrections (48) and (52) have opposite signs and their numerical values differ considerably.

VI. CONCLUSION

The calculation of the nuclear structure corrections (the nuclear size corrections) of orders $(Z\alpha)^4$ and $(Z\alpha)^5$ in the electron and muon g-factors is performed in this work on the basis of the quasipotential method. Numerical results of obtained corrections (32), (46), (48), (52) are presented in Tables II and III. We kept the same values of the nuclear charges and nuclear charge radii as in Ref.[11]. It was demonstrated that considered corrections can be successfully derived in the framework of the quasipotential method used earlier in Refs.[4, 5] for the study the radiative and recoil effects of orders $O(\alpha^2)$, $O(\alpha^3)$. The matrix element of electromagnetic current operator is obtained in new form (20) which is convenient for the calculation of different quantum electrodynamical corrections of high order over α and m_1/m_2 by means of such systems of analytical calculations as the Form, Reduce etc. Obtained results for the corrections of order $(Z\alpha)^4$ coincide with the calculations in Refs. [13, 16]. New recoil contributions $(Z\alpha)^4 m_1/m_2$, $(Z\alpha)^4(m_1/m_2)^2$ and the correction of order $(Z\alpha)^5$ in the g-factor of the bound lepton are numerically less important to the present than the leading order contribution. Their role can be significantly increased with the growth of experimental accuracy for the hydrogenic ions with sufficiently high Z. All obtained analytical relations and numerical results for the correction Δg^{str} in the $1S$ state can be extended to other S states of simple atoms. It should be pointed out that for muonic hydrogenic atoms the role of the nuclear structure corrections of orders $(Z\alpha)^4$ and $(Z\alpha)^5$ including recoil effects in the muon g-factor increases by some orders of the magnitude as compared with the g-factor of the bound electron. The theoretical expression for the gyromagnetic factor of the bound muon in the hydrogen-like atom accounting relativistic, radiative, recoil and structure corrections of the fourth order has the form [15]:

$$g_\mu(H) = 2\Big\{1 - \frac{(Z\alpha)^2}{3}\left[1-\frac{3}{2}\frac{m_\mu}{m_N}\right] - \frac{(1+Z)(Z\alpha)^2}{2}\left(\frac{m_\mu}{m_N}\right)^2 + \frac{\alpha(Z\alpha)^2}{2\pi}\left[\frac{1}{6}-\frac{1}{3}\frac{m_\mu}{m_N}\right] - \tag{58}$$

$$-\frac{(Z\alpha)^4}{12} - \frac{2\alpha(Z\alpha)^2}{\pi}\left[\frac{2}{9}\ln\left(\frac{2Z\alpha m_\mu}{m_e}\right) - \frac{5}{27} + \frac{m_e^2}{2(Z\alpha m_\mu)^2} + O(m_\mu/m_N)\right] + \frac{4}{3}m_\mu^2 R_N^2 (Z\alpha)^4\Big\}.$$

In the case of the muonic hydrogen numerical value of the nuclear structure correction of order $(Z\alpha)^4$ is equal 1.11×10^{-9} whereas the total value of the corrections to the gyromagnetic factor $g_\mu(H)$ following from Eq.(58) is $(1 - 15490 \times 10^{-9})$. So, an enlargement of experimental region to the g-factor of the bound muon in the muonic hydrogenic atoms which is studied in this work, and an enhancement of experimental accuracy can lead to more precise determination of several fundamental physical constants such as the proton charge radius and the muon-to-ptoton mass ratio.

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